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Abstract

We provide a new proof of a Kramers' type law for self-stabilizing diffusion. These diffusions correspond to the hydrodynamical limit of a mean-field system of particles and may be seen as the probabilistic interpretation of the granular media equation. We use the same hypotheses as the ones used in the work "Large deviations and a Kramers' type law for self-stabilizing diffusions" by Herrmann, Imkeller and Peithmann in which the authors obtain a first proof of the statement.

Key words and phrases: Self-stabilizing diffusion ; Exit time ; Large deviations ; Coupling method

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1 Introduction

In their remarkable work "Large deviations and a Kramers' type law for self-stabilizing diffusions", Herrmann, Imkeller and Peithmann establish large deviation results and solve the exit problem of the so-called self-stabilizing diffusion. This consists of the following model.

$$X_t^\epsilon = X_0 + \int_0^t V(X_s^\epsilon)ds - \int_0^t \int_{\mathbb{R}^d} \Phi\left(X_s^\epsilon - x\right) du_s^\epsilon(x)ds + \sqrt{\epsilon}W_t\,. \tag{1}$$

In this equation, V and Φ denote vector fields on $\mathbb{R}^d$; $(W_t)_{t\geq 0}$ is a d-dimensional Wiener process ; du_t^ϵ denotes the law of the random variable X_t^ϵ and X_0 is a deterministic real. We remark that the own law of the process intervenes in the drift which explains the term *self-stabilizing*.

Equation (1) corresponds to the hydrodynamical limit of a mean-field system of particles.

$$Z_t^{\epsilon,i,N} = X_0 + \sqrt{\epsilon} W_t^i - \int_0^t V\left(Z_s^{\epsilon,i,N}\right) ds - \frac{1}{N}\sum_{j=1}^N \int_0^t \Phi\left(Z_s^{\epsilon,i,N} - Z_s^{\epsilon,j,N}\right) ds$$

for all $1 \leq i \leq N$. Here, the W^i are independent Brownian motions and $W^1 = W$. See [Szn91].

In [HIP08], the authors consider an open domain $\mathcal{D}$ satisfying some hypotheses and they study the limit as ϵ goes to 0 of $\epsilon \log\left\{\mathbb{E}\left[\tau_{\mathcal{D}}(\epsilon)\right]\right\}$ where $\tau_{\mathcal{D}}(\epsilon)$ is defined as the first exit time of X^ϵ from the domain $\mathcal{D}$:

$$\tau_{\mathcal{D}}(\epsilon) := \inf\left\{t \geq 0 \ \mid \ X_t^\epsilon \notin \mathcal{D}\right\}.$$

More precisely, they obtain the limit

$$\lim_{\epsilon\to 0} \mathbb{P}\left\{e^{\frac{1}{\epsilon}\left(\overline{Q_\infty}-\xi\right)} < \tau_{\mathcal{D}}(\epsilon) < e^{\frac{1}{\epsilon}\left(\overline{Q_\infty}+\xi\right)}\right\} = 1\,,$$

for any $\xi > 0$. Here, $\overline{Q_\infty}$ denotes the exit cost of the domain $\mathcal{D}$:

$$\overline{Q_\infty} := \inf_{z\in\partial D}\inf_{T>0}\inf_{\varphi\in H_z^1} \frac{1}{2}\int_0^T \left|\left|\dot{\varphi}_t - V(\varphi_t) + \Phi(\varphi_t - x_{stable})\right|\right|^2 dt\,. \tag{2}$$

The set H_z^1 denotes the space of absolutely continuous functions f such that $f(0) = x_{stable}$ and $f(T) = z$. And, x_{stable} is the unique point in which the vector field V is equal to 0.

In a gradient case, we simply have $\overline{Q_\infty} := 2\inf_{z\in\partial D}\left(W(z) - W(x_{stable})\right)$, where the potential W is defined by $\nabla W(x) = V(x) - \Phi(x - x_{stable})$.

To obtain their result, they reconstruct Freidlin-Wentzell theory to the self-stabilizing diffusion. More recently, we published a work ([Tug12]) which proves the same result (albeit only in the gradient case) by using a different method. We solve the exit problem of the first particle $Z^{\epsilon,1,N}$ and we use a coupling method between $Z^{\epsilon,1,N}$ and X^ϵ. The proof is more natural and intuitive but is more technical.

The aim of this paper is to provide a much simpler method to obtain the result. For a complete review of Freidlin-Wentzell theory, see [DZ98, FW98].

First, we give the assumptions of the paper, that are the same as the ones in [HIP08] (see pages 1383 and 1406). Then, we remind the reader the main result of the paper that concerns the exit time. We have chosen not to deal with the exit location because we do not have improvement of this question. In a third section, we provide the proof of the theorem.

2 Assumptions and notations

In this work, we take the same hypotheses than the ones in [HIP08].

Assumption (A): *We say that the coefficients V and F satisfy the set of assumptions (A) if*

(A-1) *The coefficients V and Φ are locally Lipschitz, that is, for each $R > 0$ there exists $K_R > 0$ such that $||V(x) - V(y)|| + ||\Phi(x) - \Phi(y)|| \leq K_R ||x - y||$, for $x, y \in B_R(0) := \left\{z \in \mathbb{R}^d \ : \ ||z|| < R\right\}$.*

(A-2) *The interaction function Φ is rotationally invariant, that is, there exists a function ϕ from $[0; +\infty[$ to $[0; +\infty[$ such that $\Phi(x) = \frac{x}{||x||}\phi(||x||)$, $x \neq 0$.*

(A-3) *The function ϕ is convex and $\phi(0) = 0$.*

(A-4) *The function Φ grows at most polynomially: there exists $K > 0$ and $r \in \mathbb{N}$ such that $||\Phi(x) - \Phi(y)|| \leq ||x - y|| \left(K + ||x||^r + ||y||^r\right)$, $x, y \in \mathbb{R}^d$.*

(A-5) *The function V is continuously differentiable.*

(A-6) *The vector field V is convex: let $DV(x)$ denote the Jacobian of V. We assume that there exists $K_V > 0$ such that $\langle h\,;\, DV(x)h\rangle \leq -K_V$, for $h \in \mathbb{R}^d$ such that $||h|| = 1$ and $x \in \mathbb{R}^d$.*

(A-7) *We assume that the unique point in which the vector field is equal to 0 is x_{stable}.*

Under these seven assumptions, here exists a positive integer n_0 such that $\sup_{x\in\mathcal{K}} |\Phi * \mu(x) - \Phi(x - x_{stable})| \leq K\left(M_1, \cdots, M_{n_0}\right)$, K being a continuous function such that $K(0, \cdots, 0) = 0$ and $M_p := \int_{\mathbb{R}^d} ||y - x_{stable}||^p \mu(dy)$.

We now present the definition of what we denote as "stable by".

Definition 2.1. *Let k be any positive integer. Let $\mathcal{G}$ be a subset of $\mathbb{R}^k$ and let U be a vector field from $\mathbb{R}^k$ to $\mathbb{R}^k$ which satisfies the set of assumptions (A). For all $x \in \mathbb{R}^k$, we consider the dynamical system $\psi_t(x) = x + \int_0^t U\left(\psi_s(x)\right) ds$. We say that the domain $\mathcal{G}$ is stable by U if the orbit $\{\psi_t(x)\,;\, t \in \mathbb{R}_+\}$ is included in $\mathcal{G}$ for all $x \in \mathcal{G}$.*

Hypothesis 2.2. *We consider the dynamical system $\varphi_t = X_0 + \int_0^t V\left(\varphi_s\right) ds$ where X_0 is introduced in (1). The orbit $\{\varphi_t\ ;\ t > 0\}$ is included in $\mathcal{D}$.*

Hypothesis 2.3. *The open domain $\mathcal{D}$ is stable by $V - \Phi(. - x_{stable})$.*

Definition 2.4. *$\mathbb{B}_\kappa^\infty$ denotes the set of all the probability measures μ on $\mathbb{R}^d$ satisfying $\int_{\mathbb{R}^d} ||x - x_{stable}||^{2n} \mu(dx) \leq \kappa^{2n}$.*

We now give the main result of the current work.

Theorem: *We consider vector fields V and Φ which satisfy the set of assumptions (A). Under Assumptions 2.2–2.3, for all $\xi > 0$, we have the limit:*

$$\lim_{\epsilon\to 0} \mathbb{P}\left\{e^{\frac{1}{\epsilon}\left(\overline{Q_\infty} - \xi\right)} < \tau_{\mathcal{D}}(\epsilon) < e^{\frac{1}{\epsilon}\left(\overline{Q_\infty} + \xi\right)}\right\} = 1\,.$$

Let us notice that Herrmann, Imkeller and Peithmann assume a stronger hypothesis than Hypothesis 2.3. Indeed, in our work, the domain $\mathcal{D}$ is not assumed to be stable by V.

3 Proof of Theorem

For the sake of the reading, we assume $x_{stable} = 0$ in this section. Let us note that there is no loss of generality.

3.1 Control of the moments

We now establish an important result about the moments of X^ϵ. Indeed, since these moments intervene in the drift, the asymptotic behaviour (deterministic) of the law u_t^ϵ is related to the asymptotic behaviour (probabilistic) of the trajectories.

Property 3.1. 1. *The* $2n$*th moment is uniformly bounded:*

$$\sup_{t\in\mathbb{R}_+}\mathbb{E}\left\{||X_t^\epsilon||^{2n}\right\}\leq\max\left\{||X_0||^{2n}\;;\;\left(\frac{2n-1}{2K_V}\right)^n\epsilon^n\right\}\,. \tag{3}$$

2. *For all* $\kappa>0$ *and* $\epsilon>0$*, we introduce the* deterministic *time*

$$T_\kappa(\epsilon):=\min\left\{t\geq 0\;\middle|\;\mathbb{E}\left\{||X_t^\epsilon||^{2n}\right\}\leq\kappa^{2n}\right\}\,.$$

For $\epsilon<\frac{\kappa^2K_V}{2n-1}$*, we have the inequality:* $T_\kappa(\epsilon)\leq\frac{1}{nK_V\kappa^{2n}}\,||X_0||^{2n}$*.*
3. *Moreover, for all* $t\geq T_\kappa(\epsilon)$*,* $\mathbb{E}\left\{||X_t^\epsilon||^{2n}\right\}\leq\kappa^{2n}$*.*

Proof. We put $\xi_\epsilon(t):=\mathbb{E}\left\{||X_t^\epsilon||^{2n}\right\}$. We apply the Itô formula, we integrate, we take the expectation then we take the derivative. We obtain:

$$\begin{aligned}\xi_\epsilon'(t)=&2n\mathbb{E}\left\{||X_t^\epsilon||^{2n-2}\left\langle X_t^\epsilon\,;\,V\left(X_t^\epsilon\right)\right\rangle\right\}-2n\mathbb{E}\left\{||X_t^\epsilon||^{2n-2}\left\langle X_t^\epsilon\,;\,\Phi*u_t^\epsilon\left(X_t^\epsilon\right)\right\rangle\right\}\\&+n(2n-1)\epsilon\mathbb{E}\left\{||X_t^\epsilon||^{2n-2}\right\}=:2n\left(a_\epsilon(t)+b_\epsilon(t)\right)+c_\epsilon(t)\,.\end{aligned}$$

By definition, the second term $b_\epsilon(t)$ can be written as

$$b_\epsilon(t)=\mathbb{E}\left[||X_t^\epsilon||^{2n-2}\left\langle X_t^\epsilon\,;\,\Phi\left(X_t^\epsilon-Y_t^\epsilon\right)\right\rangle\right]$$

where Y^ϵ is a solution of (1) independent from X^ϵ. We can exchange X^ϵ and Y^ϵ. Thereby, by using the assumptions, we get:

$$\begin{aligned}b_\epsilon(t)&=\mathbb{E}\left\{\frac{\phi\left(||X_t^\epsilon-Y_t^\epsilon||\right)}{||X_t^\epsilon-Y_t^\epsilon||}\left\langle||X_t^\epsilon||^{2n-2}X_t^\epsilon\,;\,X_t^\epsilon-Y_t^\epsilon\right\rangle\right\}\\&=\frac{1}{2}\mathbb{E}\left\{\frac{\phi\left(||X_t^\epsilon-Y_t^\epsilon||\right)}{||X_t^\epsilon-Y_t^\epsilon||}\left\langle X_t^\epsilon\,||X_t^\epsilon||^{2n-2}-Y_t^\epsilon\,||Y_t^\epsilon||^{2n-2}\,;\,X_t^\epsilon-Y_t^\epsilon\right\rangle\right\}\,.\end{aligned}$$

This last term is nonnegative. Indeed, the Cauchy-Schwarz inequality implies

$$\left\langle x\,||x||^{2n-2}-y\,||y||^{2n-2}\,;\,x-y\right\rangle\geq\left(||x||^{2n-1}-||y||^{2n-1}\right)\left(||x||-||y||\right)\geq 0$$

for all $x,y\in\mathbb{R}^d$. Therefore, we obtain $b_\epsilon(t)\geq 0$.
Moreover, the convexity assumption implies

$$a_\epsilon(t)=\mathbb{E}\left\{||X_t^\epsilon||^{2n-2}\left\langle X_t^\epsilon\,;\,V\left(X_t^\epsilon\right)\right\rangle\right]\leq -K_V\mathbb{E}\left\{||X_t^\epsilon||^{2n}\right\}=-K_V\xi_\epsilon(t)\,.$$

Hence, by using Jensen inequality, we deduce $c_\epsilon(t) \leq n(2n-1)\epsilon\xi_\epsilon(t)^{1-\frac{1}{2n}}$. By combining results on $a_\epsilon(t)$, $b_\epsilon(t)$ and $c_\epsilon(t)$, we obtain

$$\xi_\epsilon'(t) \leq -2nK_V\xi_\epsilon(t)^{1-\frac{1}{n}}\left\{\xi_\epsilon(t)^{\frac{1}{n}} - \frac{(2n-1)\epsilon}{2K_V}\right\}. \tag{4}$$

The statements of the lemma are obvious consequences of Inequality (4).

□

This means that the self-stabilizing process tends to be trapped in a ball with center $0 = x_{stable}$.

3.2 Probability of exiting before $T_\kappa(\epsilon)$

In this paragraph, we give the following result.

Property 3.2. *We have the limit:* $\lim_{\epsilon\to 0}\mathbb{P}\left(\tau_{\mathcal{D}}(\epsilon) < T_\kappa(\epsilon)\right) = 0$ *for any* $\kappa > 0$.

We skip the proof but the ideas are the following. For all $\delta > 0$, we introduce

$$\tau_\delta(\epsilon) := \inf\left\{t > 0 \;:\; ||X_t^\epsilon - \varphi_t|| > \delta\right\},$$

where we remind the reader that $\varphi_t = X_0 + \int_0^t V(\varphi_s)\,ds$. Thus, for any $T > 0$, the following limit is an easy and classical result: $\lim_{\epsilon\to 0}\mathbb{P}\left(\tau_\delta(\epsilon) < T\right) = 0$. However, here, we consider the interval $[0;T_\kappa(\epsilon)]$ which depends on ϵ. But, we have uniformly bounded $T_\kappa(\epsilon)$. Indeed, we have

$$\mathbb{P}\left(\tau_\delta(\epsilon) < T_\kappa(\epsilon)\right) \leq \mathbb{P}\left(\tau_\delta(\epsilon) < \frac{1}{nK_V\kappa^{2n}}\,||X_0||^{2n}\right),$$

which goes to 0 as the noise elapses.

Due to hypothesis 2.2, we have $\{\varphi_t \;:\; t > 0\} \subset \mathcal{D}$. Consequently, for any $\kappa > 0$, we obtain the limit $\lim_{\epsilon\to 0}\mathbb{P}\left(\tau_{\mathcal{D}}(\epsilon) < T_\kappa(\epsilon)\right) = 0$.

3.3 Coupling result

Let $\mathcal{K}$ be a compact domain which contains the open set $\mathcal{D}$.

We have proven that the diffusion does not exit the domain $\mathcal{D}$ before the time $T_\kappa(\epsilon)$. Now, we study the exit of the diffusion from the domain after the time $T_\kappa(\epsilon)$. To do so, we use the following fact: $\sup_{t\geq T_\kappa(\epsilon)}\mathbb{E}\left\{||X_t^\epsilon||^{2n}\right\} \leq \kappa^{2n}$. Since this inequality holds for any $\kappa > 0$, we deduce that the drift $V - \Phi * u_t^\epsilon$ is close to the vector field $V - \Phi * \delta_0 = V - \Phi$. Consequently, we consider the following diffusion defined for $t \geq T_\kappa(\epsilon)$:

$$Y_t^\epsilon = X_{T_\kappa(\epsilon)} + \sqrt{\epsilon}\left(W_t - W_{T_\kappa(\epsilon)}\right) + \int_{T_\kappa(\epsilon)}^t V(Y_s^\epsilon)ds - \int_{T_\kappa(\epsilon)}^t \Phi\left(Y_s^\epsilon\right)ds, \tag{5}$$

if $X_{T_\kappa(\epsilon)} \in \mathcal{K}$ and $Y_t^\epsilon := X_t^\epsilon$ otherwise. We introduce the two following exit time: $\tau_{\mathcal{K}}(\epsilon) := \inf\left\{t > T_\kappa(\epsilon) \;:\; X_t^\epsilon \notin \mathcal{K}\right\}$ and $\tau'_{\mathcal{K},\kappa}(\epsilon) := \inf\left\{t > T_\kappa(\epsilon) \;:\; Y_t^\epsilon \notin \mathcal{K}\right\}$.

We introduce the stopping time: $\mathcal{T}_{\mathcal{K},\kappa}(\epsilon) := \min\left\{\tau_{\mathcal{K}}(\epsilon); \tau'_{\mathcal{K}}(\epsilon)\right\}$. The following result tells us that the two diffusions are close on $[T_\kappa(\epsilon)\,;\,\mathcal{T}_{\mathcal{K},\kappa}(\epsilon)]$.

Theorem 3.3. *There exists κ_0 such that for all $\kappa < \kappa_0$, there exists $\epsilon_0(\kappa) > 0$ such that $\mathbb{P}\left\{\sup_{T_\kappa(\epsilon)\leq t\leq \mathcal{T}_{\mathcal{K},\kappa}(\epsilon)} ||X_t^\epsilon - Y_t^\epsilon|| \geq r(\kappa)\right\} \leq r(\kappa)$ for all $\epsilon < \epsilon_0(\kappa)$. Here, r is a positive and increasing function such that $r(0) = 0$.*

Proof. **Step 1.** We introduce the vector fields $H_\infty(x) := V(x) - \Phi(x)$ and $H_t(x) := V(x) - \Phi * u_t^\epsilon(x)$. The assumptions on V and Φ imply $DH_t(x) \leq -K_V < 0$. From now on, we put $\xi_\epsilon(t) := ||X_t^\epsilon - Y_t^\epsilon||$. If $X_{T_\kappa}^\epsilon, Y_{T_\kappa}^\epsilon \in \mathcal{K}$ then, for all $T_\kappa \leq t \leq \mathcal{T}_\kappa(\epsilon)$, we have:

$$
\begin{aligned}
\frac{d}{dt}\left(\xi_\epsilon(t)\right)^2 =& -2\left\langle H_t\left(X_t^\epsilon\right) - H_\infty\left(Y_t^\epsilon\right) ; X_t^\epsilon - Y_t^\epsilon\right\rangle \\
=& -2\left\langle H_t\left(X_t^\epsilon\right) - H_t\left(Y_t^\epsilon\right) ; X_t^\epsilon - Y_t^\epsilon\right\rangle \\
& -2\left\langle \Phi * u_t^\epsilon\left(Y_t^\epsilon\right) - \Phi\left(Y_t^\epsilon\right) ; X_t^\epsilon - Y_t^\epsilon\right\rangle \\
\leq& -2K_V\left(\xi_\epsilon(t)\right)^2 + 2\xi_\epsilon(t) f_{\mathcal{K}}(\kappa)\,, \qquad (6)
\end{aligned}
$$

where we set $f_{\mathcal{K}}(\kappa) := \sup_{\mu_1\in\mathbb{B}_\kappa^\infty}\sup_{x\in\mathcal{K}} ||\Phi*\mu_1(x) - \Phi(x)|| =: K_V r(\kappa)^{\frac{3}{2}}$. Inequality (6) directly implies $\sup_{T_\kappa\leq t\leq\mathcal{T}_\kappa(\epsilon)} ||X_t^\epsilon - Y_t^\epsilon||^2 \leq r(\kappa)^3$ which yields $\mathbb{E}\left\{\sup_{T_\kappa(\epsilon)\leq t\leq\mathcal{T}_\kappa(\epsilon)} ||X_t^\epsilon - Y_t^\epsilon||^2\right\} \leq r(\kappa)^3$. The claim thus follows from the Markov inequality. □

3.4 Proof

Step 1. Let $\kappa > 0$. We can easily prove (by proceeding like in [Tug12]) that there exist two families of domains $\left(\mathcal{D}_{i,\kappa}\right)_{\kappa>0}$ and $\left(\mathcal{D}_{e,\kappa}\right)_{\kappa>0}$ such that

- $\mathcal{D}_{i,\kappa} \subset \mathcal{D} \subset \mathcal{D}_{e,\kappa}$.
- $\mathcal{D}_{i,\kappa}$ and $\mathcal{D}_{e,\kappa}$ are stable by $V - \Phi$. The terminology "stable by" has been introduced in Definition 2.1.
- $\sup_{z\in\partial\mathcal{D}_{i,\kappa}} \mathrm{d}\left(z\,;\mathcal{D}^c\right) + \sup_{z\in\partial\mathcal{D}_{e,\kappa}} \mathrm{d}\left(z\,;\mathcal{D}\right)$ tends to 0 when κ goes to 0.
- $\inf_{z\in\partial\mathcal{D}_{i,\kappa}} \mathrm{d}\left(z\,;\mathcal{D}^c\right) = \inf_{z\in\partial\mathcal{D}_{e,\kappa}} \mathrm{d}\left(z\,;\mathcal{D}\right) = r(\kappa)$.

Step 2. By $\tau_{i,\kappa}(\epsilon)$ (resp. $\tau_{e,\kappa}(\epsilon)$), we denote the first exit time of Y^ϵ from $\mathcal{D}_{i,\kappa}$ (resp. $\mathcal{D}_{e,\kappa}$).

Step 3. We prove here the upper bound:

$$
\begin{aligned}
\mathbb{P}\left\{\tau_{\mathcal{D}}(\epsilon) \geq e^{\frac{\overline{Q_\infty}+\xi}{\epsilon}}\right\} =& \mathbb{P}\left\{\tau_{\mathcal{D}}(\epsilon) \geq e^{\frac{\overline{Q_\infty}+\xi}{\epsilon}} ; \tau_{e,\kappa}(\epsilon) \geq e^{\frac{\overline{Q_\infty}+\xi}{\epsilon}}\right\} \\
& + \mathbb{P}\left\{\tau_{\mathcal{D}}(\epsilon) \geq e^{\frac{\overline{Q_\infty}+\xi}{\epsilon}} ; \tau_{e,\kappa}(\epsilon) < e^{\frac{\overline{Q_\infty}+\xi}{\epsilon}}\right\} \\
\leq& \mathbb{P}\left\{\tau_{e,\kappa}(\epsilon) \geq e^{\frac{\overline{Q_\infty}+\xi}{\epsilon}}\right\} \\
& + \mathbb{P}\left\{\tau_{\mathcal{D}}(\epsilon) \geq e^{\frac{\overline{Q_\infty}+\xi}{\epsilon}} ; \tau_{e,\kappa}(\epsilon) < e^{\frac{\overline{Q_\infty}+\xi}{\epsilon}}\right\} \\
& =: a_\kappa(\epsilon) + b_\kappa(\epsilon)\,.
\end{aligned}
$$

Step 3.1. By classical results in Freidlin-Wentzell theory, there exists $\kappa_1 > 0$ such that for all $0 < \kappa < \kappa_1$, we have: $\lim_{\epsilon\to 0}\mathbb{P}\left\{\tau_{e,\kappa}(\epsilon) < \exp\left[\frac{1}{\epsilon}\left(\overline{Q_\infty} + \xi\right)\right]\right\} = 0$. Therefore, the first term $a_\kappa(\epsilon)$ tends to 0 as ϵ goes to 0.
Step 3.2. Let us look at the second term $b_\kappa(\epsilon)$. For κ sufficiently small, we have $\mathcal{D}_{e,\kappa} \subset \mathcal{K}$. Consequently, we have:

$$\mathbb{P}\left\{\tau(\epsilon) \geq e^{\frac{\overline{Q_\infty}+\xi}{\epsilon}}\,;\, \tau_{e,\kappa}(\epsilon) \leq e^{\frac{\overline{Q_\infty}+\xi}{\epsilon}}\right\}$$

$$\leq \mathbb{P}\left\{\left|\left|X^\epsilon_{\tau_{e,\kappa}(\epsilon)} - Y^\epsilon_{\tau_{e,\kappa}(\epsilon)}\right|\right| \geq r(\kappa)\right\} \leq \mathbb{P}\left\{\sup_{T_\kappa(\epsilon)\leq t\leq \mathcal{T}_{\mathcal{K},\kappa}(\epsilon)} ||X^\epsilon_t - Y^\epsilon_t|| \geq r(\kappa)\right\}.$$

According to Theorem 3.3, there exists $\epsilon_0 > 0$ such that the previous term is less than $r(\kappa)$ for all $\epsilon < \epsilon_0$.
Step 3.3. Let $\xi > 0$. By taking κ arbitrarily small, we obtain the upper bound $\lim_{\epsilon\to 0}\mathbb{P}\left\{\tau(\epsilon) \geq \exp\left[\frac{\overline{Q_\infty}+\xi}{\epsilon}\right]\right\} = 0$.
Step 3. Analogous arguments show that $\lim_{\epsilon\to 0}\mathbb{P}\left\{T_\kappa(\epsilon) \leq \tau(\epsilon) \leq e^{\frac{\overline{Q_\infty}-\xi}{\epsilon}}\right\} = 0$. However, we have $\lim_{\epsilon\to 0}\mathbb{P}\left\{\tau(\epsilon) \leq T_\kappa(\epsilon)\right\} = 0$. This ends the proof. □

Remark 3.4. *In the theorem, we give the exit time of the McKean-Vlasov diffusion but we could use the same technics to provide the exit time of the first particle in the mean-field system of particles. The only difference is that we would need to use the first time that the empirical measure exits from the ball* $\mathbb{B}^\infty_\kappa$ *; which is close to the arguments in [Tug12].*

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References

[DZ98] A. Dembo and O. Zeitouni: *Large deviations techniques and applications*, volume 38 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, Berlin, 2010. Corrected reprint of the second (1998) edition.

[FW98] M. I. Freidlin and A. D. Wentzell: *Random perturbations of dynamical systems*, volume 260 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, second edition, 1998. Translated from the 1979 Russian original by Joseph Szücs.

[HIP08] Samuel Herrmann, Peter Imkeller, and Dierk Peithmann. Large deviations and a Kramers' type law for self-stabilizing diffusions. *Ann. Appl. Probab.*, 18(4):1379–1423, 2008.

[Szn91] Alain-Sol Sznitman. Topics in propagation of chaos. In *École d'Été de Probabilités de Saint-Flour XIX—1989*, volume 1464 of *Lecture Notes in Math.*, pages 165–251. Springer, Berlin, 1991.

[Tug12] J. Tugaut. Exit problem of McKean-Vlasov diffusions in convex landscapes. *Electronic Journal of Probability*, Vol. 17 (2012), no. 76, 1–26.